Visualizing the kinematics of relativistic wave packets

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Abstract. This article presents visualizations of some solutions of the time-dependent free Dirac equation. They illustrate some of the strange phenomena which are caused by the interference of positive- and negative-energy waves. We discuss, in particular, the conditions for the occurrence of effects like the *Zitterbewegung*, the opposite direction of momentum and velocity in negative-energy wave packets, and the superluminal propagation of the wave packet's local maxima.

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1. Introduction

The Dirac equation is the fundamental equation for relativistic quantum mechanics. Among its big successes is the very accurate description of the energy levels of the hydrogen atom. In the historical development, however, the occurrence of several paradoxa has made it difficult to find an appropriate interpretation. A prominent place among these phenomena is taken by Zitterbewegung, which was originally discovered by E. Schrödinger [1] and is still subject of recent publications (for example, [2, 3]).

For free particles, Zitterbewegung and similar effects can be discussed away by restricting the consideration to positive (or negative) kinetic energies (particles, resp. antiparticles). In the presence of external fields, this can only be done in an approximate sense, because in this case the sign of the kinetic energy (described by the free Dirac operator) is not a constant of motion. For fields vanishing at infinity, one can at least show that Zitterbewegung vanishes as $|t| \to \infty$ [4], so that a particle-antiparticle interpretation can be maintained at least asymptotically.

Irrespective of any chosen interpretation, Zitterbewegung is a mathematical phenomenon shown by some solutions of the Dirac equation. A precise understanding of this effect and the conditions of its occurrence is necessary, for example, in a proof of the asymptotic completeness of the relativistic scattering operator. Moreover, it is almost unavoidable to stumble across this and similar strange phenomena, when one attempts a numerical solution of the Dirac equation.

The numerical simulation of the Schrödinger or Dirac equation and the visualization of its solutions has become an important part of quantum-mechanical education on all levels [5, 7, 6]. Indeed, the possibilities of modern computers might suggest the approach to the teaching of quantum mechanics chosen, for example, in the books [8, 9]. Here one uses numerous visualizations of wave functions in many different situations in order to build some intuitive understanding for the behavior of these solutions and in order to motivate the theoretical investigation of this behavior.

In case of the Dirac equation, this approach would quickly lead us to the necessity of discussing the unexpected behavior that occurs already for very innocent-looking initial conditions. We believe that these phenomena should not be hidden from students, because they stir enough curiosity to motivate a detailed theoretical discussion of the role of negative energy solutions and their interpretation in terms of antiparticles. This could lead to a deeper understanding of the foundations of quantum electrodynamics whose state space is built from tensor products of one-particle and one-antiparticle Hilbert spaces [10].

In this article we want to show some visualizations that seem to be of interest even for the expert in the field. In order to be concise, we concentrate on the kinematical effects of the free Dirac equation. It is the behavior in an external field that leads to the interpretation of the negative-energy states as being unitarily equivalent to antiparticle states (via a charge conjugation). Some interesting visualizations of relativistic wave packets in external fields will be discussed in a forthcoming article.

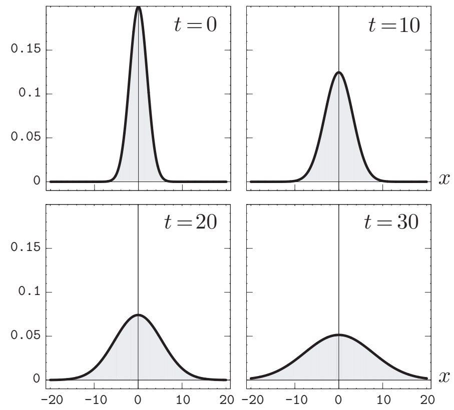

Figure 1. Spreading of a Gaussian wave packet according to the Schrödinger equation.

The printed article contains only some black-and-white images. It is much more instructive to watch movies showing the whole time evolution. Therefore, this article is supplemented by some computer-generated animations (QuickTime movies) that can be accessed via the internet [11]. These movies show both components of the solution with a color code for the phase of complex numbers. Thus they reveal much more information than the simple black-and-white reproductions of the position probability density in this article. (The colors show, in particular, the direction and the size of the momentum.) The book [9] is accompanied by a CD-ROM with a large collection of similar movies, animations, and simulations, some examples can be seen on the web [12].

2. An example

A canonical set of initial conditions for the time-dependent Schrödinger or Dirac equations is given by the set of Gaussian wave packets. They describe more or less localized quantum states for which the product of the uncertainties in position and momentum is minimal. On the other hand, the set of Gaussian initial conditions covers most cases of practical interest, because any wave packet can be approximated by a superposition of a finite number of Gaussian states.

The motion of Gaussian wave packets according to the one-dimensional free Schrödinger equation shows little surprises. The example in Figure 1 shows a nonrelativistic Gaussian wave packet with average momentum zero. Initially, the wave packet is well localized, but it spreads during the time evolution. As the wave packet gets smeared out, its height decreases, because its norm as a square-integrable function must remain constant. According to the Schrödinger equation, a Gaussian wave packet remains a Gaussian function for all times.

The average position $\langle \mathbf{x} \rangle$ and the average momentum $\langle \mathbf{p} \rangle$ of a free Schrödinger wave packet obey the rules of classical mechanics. Moreover, the spreading of the wave packet is independent of the average velocity of the wave packet. This spreading of the position distribution would also be observed for a cloud of classical particles whose density in

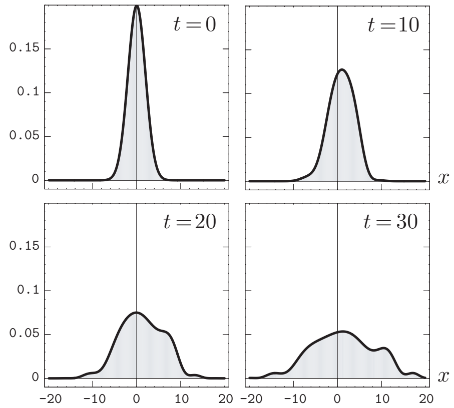

Figure 2. Time evolution of a Gaussian initial wave packet according to the Dirac equation.

position space is a Gaussian function, provided that the momenta of the particles also have a Gaussian distribution.

Figure 2 shows a numerical solution of the one-dimensional free Dirac equation. It has the same Gaussian initial distribution as the nonrelativistic wave packet in Figure 1, yet its behavior is quite different. The relativistic wave packet wiggles back and forth, becomes non-Gaussian for $t \neq 0$, and soon develops characteristic ripples.

This result is so strange that anybody with some experience with quantum mechanics (but not with the Dirac equation) would first assume that the numerical method is at fault. This is a good example supporting the argument that the numerical solution of an equation is rarely sufficient to understand a phenomenon. It motivates a more careful theoretical analysis in order to understand the origin of this strange behavior.

It should be mentioned that the Dirac equation does have solutions that behave similar to that of the nonrelativistic wave packet in Figure 1. These solutions can be obtained, for example, from Gaussian wave packets by a projection onto the subspace of positive energies. The behavior of these "reasonable solutions" is not the subject of this article (see [9] for more details).

3. The Dirac equation

In this article, we discuss the time-dependent free Dirac equation in one space dimension. We write it as an evolution equation in "Schrödinger form"

$$\mathrm{i}\hbar \frac{d}{dt}\psi(x,t) = H_0\,\psi(x,t), \qquad \psi(x,0) = \psi_0(x). \tag{1}$$

At this stage, and from a pedagogical point of view, we prefer this notation to the covariant form, because it is more similar to the nonrelativistic case and it allows us to use the standard methods of quantum mechanics. The free Dirac Hamiltonian H_0 is the matrix-differential operator

$$H_0 = c\,\sigma_1\,p + \sigma_3\,mc^2, \tag{2}$$

where σ_1 and σ_3 are the famous Pauli matrices, and $p = -\mathrm{i}\,\hbar\,d/dx$. Note that the Dirac equation in one space-dimension has just two components instead of four, because there is no spin-flip in one dimension. Basically, this is a consequence of the representation theory of the Poincaré group.

The expression for H_0 can be interpreted as a linearization of the relativistic energy-momentum relation

$$E = \lambda(p) = \sqrt{c^2p^2 + m^2c^4}. \tag{3}$$

The square of the Dirac operator is just given by

$$H_0^2 = c^2p^2 + m^2c^4. \tag{4}$$

For numerical computations and for the visualizations it is advantageous to use units where $\hbar = m = c = 1$. Thus the light cone in the space-time diagrams has the opening angle of 90 degrees, as usual. The dimensionless units can be obtained from the SI units by a simple scaling transformation. Hence, in the following, we use the Dirac equation in the form

$$\mathrm{i}\,\frac{\partial}{\partial t}\,\psi(x,t) = \Bigl(-\mathrm{i}\,\sigma_1\,\frac{\partial}{\partial x} + \sigma_3\Bigr)\,\psi(x,t). \tag{5}$$

Instead of σ_1 and σ_3, we could use any other pair of Pauli matrices. This would give a unitarily equivalent formulation. All images in this article would remain unchanged.

The phenomena to be discussed here also occur in higher dimensions, but the one-dimensional situation is much easier to visualize. For the Dirac Hamiltonian in three space-dimensions, Pauli matrices are not sufficient; 4×4-Dirac matrices are needed instead, see [4] for details.

4. Dirac spinors and their interpretation

A suitable state space for the solutions of the Dirac equation must consist of vector-valued functions

$$\psi(x,t) = \begin{pmatrix} \psi_1(x,t) \\ \psi_2(x,t) \end{pmatrix}, \tag{6}$$

because the operator H_0 is a two-by-two matrix. These two-component wave functions are usually called Dirac spinors. The word "spinor" might appear inappropriate. In one dimension, all magnetic fields are pure gauge fields and cannot affect the spin. Indeed, the two-dimensional structure of the state space does not describe the spin. It rather reflects the occurrence of both signs of the energy. (In three dimensions, however, the doubling of the number of spinor components is caused by the spin.)

The energy of a free particle can have both signs, because the free Dirac Hamiltonian in momentum space is the matrix

$$\mathrm{h}_0(p) = \begin{pmatrix} mc^2 & cp \\ cp & -mc^2 \end{pmatrix}, \tag{7}$$

which for each $p \in \mathbf{R}$ has the eigenvalues $\lambda(p)$ and $-\lambda(p)$.

Correspondingly, the Dirac equation has two types of plane-wave solutions, which we denote by u_{pos} and u_{neg}. For each $p \in \mathbf{R}$,

$$u_{\substack{\text{pos}\\\text{neg}}}(p; x, t) = \frac{1}{\sqrt{2\pi}}\, u_{\substack{\text{pos}\\\text{neg}}}(p)\, \mathrm{e}^{\mathrm{i}px \mp \mathrm{i}\lambda(p)t}, \tag{8}$$

where $u_{\text{pos}}(p)$ and $u_{\text{neg}}(p)$ are eigenvectors of the matrix $\mathrm{h}_0(p)$ belonging to the eigenvalues $\lambda(p)$ and $-\lambda(p)$, respectively. Hence, we have

$$H_0\, u_{\substack{\text{pos}\\\text{neg}}}(p; x, t) = \pm\lambda(p)\, u_{\substack{\text{pos}\\\text{neg}}}(p; x, t) \tag{9}$$

and therefore u_{pos} (u_{neg}) is called a plane wave with positive (negative) energy.

In order to be in line with the formalism of quantum mechanics, one usually requires that (for each t) the components of a Dirac-spinor be square-integrable,

$$\int_{-\infty}^{\infty} |\psi_j(x, t)|^2\, dx < \infty \quad \text{(for all } t \text{ and } j = 1, 2). \tag{10}$$

One should be aware that this mathematical requirement is closely related to the interpretation of the solutions. The following interpretation appears implicitly in many articles about the Dirac equation.

Suppose ψ is a normalized Dirac spinor, then

$$\int_a^b (|\psi_1(x, t)|^2 + |\psi_2(x, t)|^2)\, dx \tag{11}$$

is interpreted as the probability to find the particle in the interval $(a.b) \subset \mathbf{R}$. Here "normalized" means that the above integral from $-\infty$ to $+\infty$ is equal to 1.

Similarly, and in complete analogy to nonrelativistic quantum mechanics,

$$\int_a^b (|\hat{\psi}_1(p, t)|^2 + |\hat{\psi}_2(p, t)|^2)\, dp \tag{12}$$

is interpreted as the probability to find the momentum of the particle in the interval from a to b. (The hat denotes the Fourier transform with respect to x.) This interpretation is consistent with the choice of $p = -\mathrm{i}d/dx$ (the generator of spatial translations) as the momentum operator.

It is certainly reasonable to start an introduction to relativistic quantum mechanics with an interpretation that is as close as possible to nonrelativistic quantum mechanics. After all, we need an interpretation in order to compare mathematical results with our expectations or with the experiments. For the beginner, the interpretation above is a convenient working hypothesis because it contains only minimal assumptions and it allows one to apply the well-established mathematical formalism of quantum mechanics. But it should be made clear from the beginning, that this interpretation will have to be modified to account for new observations, in particular, in view of the behavior of negative-energy wave packets in external fields. Unfortunately, a completely satisfying interpretation cannot be reached at the one-particle level, for the following reason.

The properties of the Dirac equation with electromagnetic fields suggest to interpret solutions with positive energy as particle states and solutions with negative energies as (charge-conjugated) antiparticle states. A problem occurs in the presence of external fields that may cause transitions from particle to antiparticle states and vice versa. Hence, if we start with a particle state, we would end up with a superposition of particle and antiparticle states (Klein's paradox). This phenomenon cannot easily be interpreted in terms of a spontaneous electron-positron pair creation, because the unitarity of the Dirac time-evolution (which is true for arbitrary external fields) guarantees that the normalization of Dirac spinors remains time-independent. Hence, if there is one particle at the beginning, there will be one particle at the end. The Dirac equation is not able to describe pair production within the framework of quantum mechanics.

It is commonly believed that Kein's paradox can be understood as a pair creation process in the framework of quantum electrodynamics, but there are still ongoing discussions about this subject. One of the reasons is that the one-particle states are used to construct the Fock space of quantum electrodynamics, and therefore, to some extent, difficulties with one-particle solutions carry over to a rigorous formulation of quantum field theory. In any case, difficulties with the interpretation of the Dirac equation should not prevent us from investigating its solutions, because these difficulties provide some motivation for a deeper theoretical analysis.

Square-integrable wave packets can be obtained from the plane waves by superposition. This is the same procedure that works for the Schrödinger equation. With suitable coefficient functions, any square-integrable solution of the Dirac equation can be written in the form

$$\psi(x,t) = \int_{-\infty}^{\infty} (\hat{\psi}_{\text{pos}}(p)\, u_{\text{pos}}(p;x,t) + \hat{\psi}_{\text{neg}}(p)\, u_{\text{neg}}(p;x,t))\, dp. \tag{13}$$

The coefficient functions $\hat{\psi}_{\text{pos}}$ and $\hat{\psi}_{\text{neg}}$ can be determined from the Fourier transform $\hat{\psi}(p,0)$ of the initial function $\psi(x,0)$ by a projection onto the positive or negative energy subspace:

$$\hat{\psi}_{\substack{\text{pos}\\\text{neg}}}(p) = P_{\substack{\text{pos}\\\text{neg}}}\, \hat{\psi}(p,0) = \frac{1}{2}\Big(\mathbf{1} \pm \frac{\mathrm{h}_0(p)}{\lambda(p)}\Big)\, \hat{\psi}(p,0). \tag{14}$$

Numerical integration of (13) provides a method to compute the free time evolution of an arbitrary initial function that can be a useful alternative to a finite difference scheme. More about the mathematics of the one-dimensional Dirac equation can be found in [9].

5. Examples of relativistic kinematics

In this section, we describe and visualize three strange phenomena shown by solutions of the free Dirac equation. Some explanations will be given in the following sections.

As a first example, we compute the free time evolution of the Gaussian Dirac spinor

$$\psi(x,0) = \Big(\frac{1}{32\pi}\Big)^{1/4} \exp(-x^2/16) \begin{pmatrix}1\\1\end{pmatrix}. \tag{15}$$

A few snapshots of the solution are shown in Figure 2. This image shows the position probability density $|\psi_1(t,x)|^2 + |\psi_2(t,x)|^2$ according to the interpretation given above. Compared to the simple spreading of the nonrelativistic Gaussian, this wave packet shows a very complicated motion.

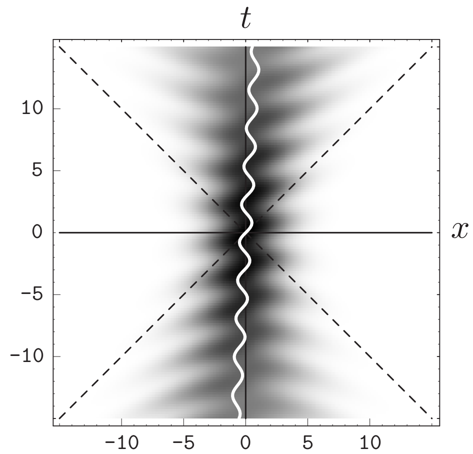

Figure 3. Space-time diagram and worldline of the position mean value for the wave packet in Figure 2.

The choice of the numerical constants in (15) is motivated by the following consideration. The Fourier transform of the wave packet (15) is

$$\hat{\psi}(p,0) = \Bigl(\frac{2}{\pi}\Bigr)^{1/4} \exp(-4p^2) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{16}$$

It describes a momentum distribution that is well localized in the interval $[-3/4, +3/4]$. This corresponds to a maximal speed less than 3/5, far below the speed of light $c = 1$. Hence, the observed effects are not caused by relativistic velocities.

Figure 3 shows a space-time diagram of this solution. Here, the density plot visualizes the position probability density as a function of the space coordinate x and the time coordinate t in dimensionless units. The white curve shows the world line of the average position, which obviously does not obey classical (relativistic) kinematics. Instead, the expectation value of the position operator performs a wiggling motion, commonly known as "Zitterbewegung" [1]. Apart from the rapid oscillation, the wave packet drifts slowly to the right, although its average momentum is zero. Nevertheless, it turns out that the momentum distribution (as determined from the Fourier transform of the wave packet) is still a conserved quantity with average momentum zero.

The second example, Figure 4 shows the free time evolution of the initial spinor

$$\psi(x,0) = \Bigl(\frac{1}{32\pi}\Bigr)^{1/4} \exp(-x^2/16 - \mathrm{i}3x/4) \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{17}$$

This spinor is very similar to (15). In position space, it is multiplied by a phase factor $\exp(-\mathrm{i}3x/4)$. In momentum space, this means a translation by 3/4. The momentum distribution belonging to (17) is therefore a Gaussian distribution centered around the average momentum 3/4. Moreover, the momentum distribution is so narrow that all momenta contributing significantly to the wave packet are positive. Nevertheless, the solution splits into two parts, and the smaller part moves to the left (that is, with a negative velocity).

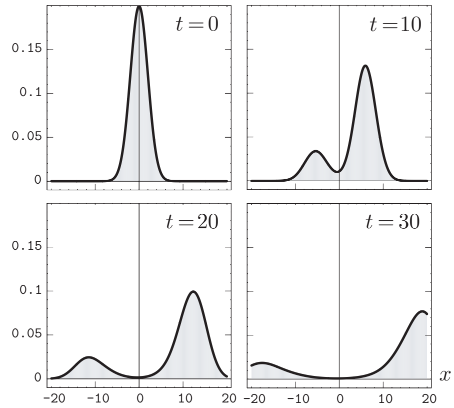

Figure 4. Time evolution of a Gaussian initial wave packet with positive momentum.

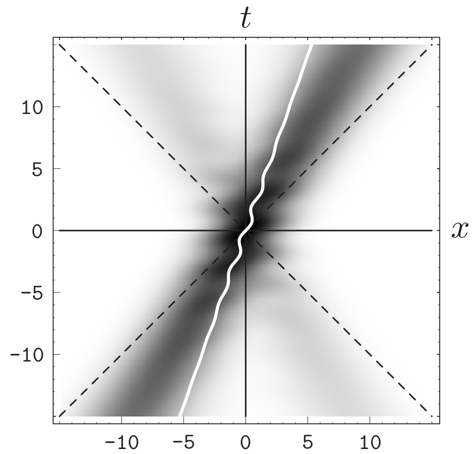

Figure 5. Space-time diagram and worldline of the position mean value for the wave packet in Figure 4.

The space-time diagram of this solution, shown in Figure 5, again shows the Zitterbewegung of the position's mean value. But this time, the oscillation quickly fades away. We see that Zitterbewegung is sustained only as long as the left-moving part and the right-moving part of the wave packet have some overlap in position space.

The third example is shown in Figure 6. It realizes a wave packet with positive velocities. The initial wave packet was obtained as a superposition of a positive- and a negative-energy part:

$$\psi(x,0) = \psi_{\text{pos}}(x) + \psi_{\text{neg}}(x), \tag{18}$$

The positive-energy part has positive momentum. In momentum space, it is defined as

$$\hat{\psi}_{\text{pos}}(x) = N P_{\text{pos}} \exp(-4(p-4/5)^2) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{19}$$

The negative-energy part has negative momentum,

$$\hat{\psi}_{\text{neg}}(x) = N P_{\text{neg}} \exp(-4(p+4/5)^2) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{20}$$

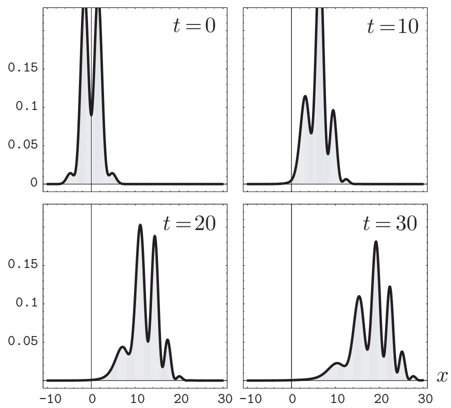

Figure 6. Time evolution of a wave packet with positive velocity.

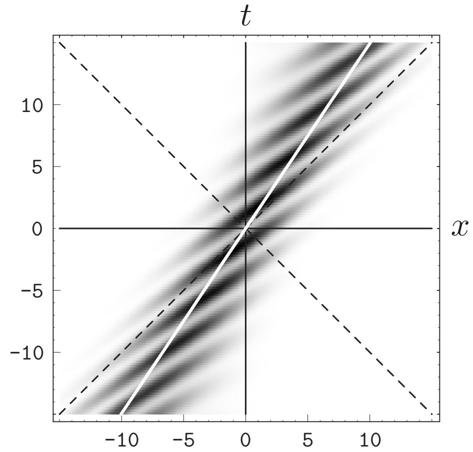

Figure 7. Space-time diagram and worldline of the position mean value for the wave packet in Figure 6.

In position space, both parts obviously move in the same direction. They interfere with each other, thereby causing the ripples in the position distribution.

Figure 7 shows a space-time diagram of this motion. There is no Zitterbewegung at all; the world line of the average position is a straight line. Interestingly, the peaks of the wave packet move with superluminal speed. This is strange because, at least in principle, local variations of the position probability density are observable. Let us now proceed with a theoretical analysis.

6. Parity and direction of motion

The average velocity of the wave packet (15) shows a slow drift to the right, which is clearly visible in Figure 3. It is remarkable that the solution is not symmetric with respect to reflections at the origin, although the initial condition is. Indeed, the Dirac

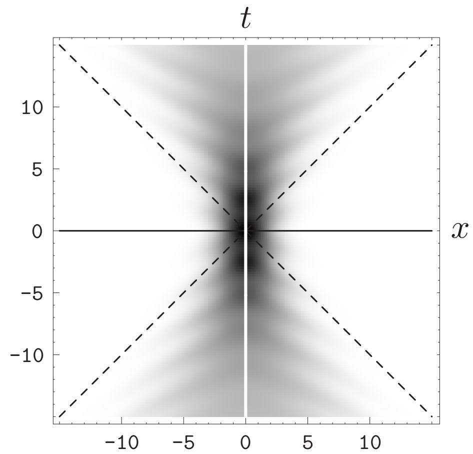

Figure 8. Space-time diagram and worldline of the position mean value for the wave packet (23).

equation is not invariant under the replacement

$$\psi(x,t) \to \psi(-x,t). \tag{21}$$

But the replacement $x \to -x$ in the wave function does not describe the physical space reflection. Like any Lorentz transformation, the space reflection has a part that acts on the components of the wave function. The correct way to describe the space reflection in the Hilbert space of the Dirac equation is the parity transform

$$P : \psi(x,t) \to \sigma_3\psi(-x,t). \tag{22}$$

Hence, the wave packet (15) is not invariant under a parity transform, $P\psi$ is not a scalar multiple of ψ.

An example of a parity invariant solution is provided by the Gaussian initial wave packet, corresponding to a particle "at rest",

$$\psi(x,0) = \Bigl(\frac{1}{4\pi}\Bigr)^{1/4} \exp(-x^2/8) \begin{pmatrix}1\\0\end{pmatrix}, \tag{23}$$

which has only an upper component and satisfies $P\psi(x,t) = \psi(x,t)$ for all times. Figure 8 shows a space-time diagram of the position probability density for this solution. The average velocity is zero, and the wave packet shows no Zitterbewegung in the expectation value—although the position probability density shows ripples similar to the first solution.

The average velocity of a wave packet is described by the classical velocity operator

$$v_{\rm cl}(p) = c^2\, p\, H_0^{-1}. \tag{24}$$

It corresponds to the definition of the velocity in terms of momentum and energy, which is familiar from classical relativistic mechanics. Note that this relation between velocity and momentum depends, in particular, on the sign of the energy. For a wave packet with negative energy, a positive momentum p thus corresponds to a negative average

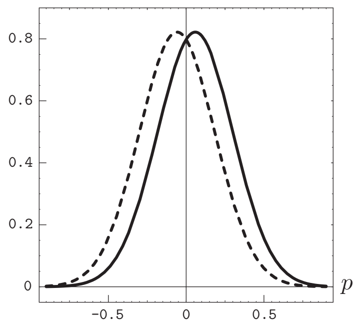

Figure 9. Momentum distributions of the positive and negative energy parts of the wave packet in Figure 9.

velocity $v_{\rm cl}$. This is also reflected by the phase velocity of the plane waves (8). The phase velocity of the plane waves is

$$v_{\rm ph} = \operatorname{sign}(E)\,\frac{\lambda(p)}{p}. \tag{25}$$

Note that the phase velocity is always faster than the velocity of light, and even tends to infinity in the limit of small momenta. This does not matter, because no information can be transmitted with phase velocity. Actually, a plane wave is spread over all of space-time and all information that is carried by a plane wave is already everywhere. The group velocity of relativistic wave packets is always slower than or equal to the velocity of light. But the sign of the phase velocity carries over to the sign of the group velocity.

For the wave packet (15), Figure 9 shows the momentum distributions of the parts with positive and negative energy, that is, the functions $|\hat{\psi}_{\rm pos}(p)|^2$ and $|\hat{\psi}_{\rm neg}(p)|^2$, according to (14). We see that the positive-energy part has its momentum distribution slightly shifted towards positive momenta, whereas the negative-energy part has a negative average momentum. For a positive-energy wave packet, a positive average momentum means a positive average velocity, as usual. But, for a negative-energy wave packet, the negative average momentum corresponds to a positive average velocity. Hence, the whole wave packet has a positive average velocity, which can be seen clearly in Figure 3.

The opposite direction of velocity and momentum in the negative-energy part of a wave packet immediately explains the behavior of the wave packet (17) in Figure 4. For this wave packet, the momentum distributions of the parts with positive and negative energy are shown in Figure 10. We see that both parts consist of positive momenta. For the smaller negative-energy part, this corresponds to a negative velocity. Hence, the wave packet in position space is a superposition of an approximately Gaussian wave packet with positive energy moving to the right, and a smaller part with negative energy moving to the left (also with a positive average momentum).

In case of the third example (18), the negative-energy part has negative momenta, as shown in Figure 11. Here the two parts are of the same size and they have opposite

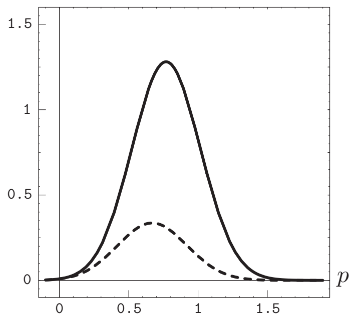

Figure 10. Momentum distributions of the positive and negative energy parts of the wave packet in Figure 4.

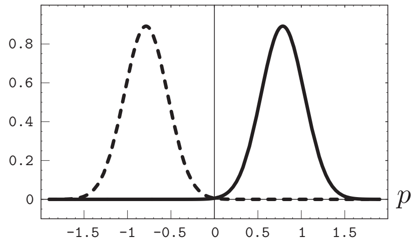

Figure 11. Momentum distributions of the positive and negative energy parts of the wave packet in Figure 6.

momenta, they have the same average velocity. Hence, the parts with positive and negative energy move together in position space. The opposite phase-directions of the two parts cause the interference effects. At the average momentum $\langle p\rangle$, the initial state contains the superposition of the plane waves $u_{\mathrm{pos}}(\langle p\rangle; x, 0) + u_{\mathrm{neg}}(-\langle p\rangle; x, 0)$. In both components of the spinor, this is proportional to a superposition of $\exp(\mathrm{i}\langle p\rangle x)$ and $\exp(-\mathrm{i}\langle p\rangle x)$. This, precisely, causes the sine-like distortions of the Gaussian position probability density visible in Figure 6.

Hence, the interference ripples have their origin in the phase interference of the parts with positive and negative energy. As the phase moves with a superluminal average velocity $v_{\mathrm{ph}} = \lambda(\langle p\rangle)/\langle p\rangle$, also the interference pattern moves with superluminal speed. Because it is just an interference pattern, it carries no information from one point in space to another. Hence, despite the fact that local variations in the position probability density are, in principle, observable, the fact that these variations move at superluminal speed is no contradiction to the special theory of relativity. A classical example for this effect is the apparent motion of Moiré patterns. If two periodic patterns with slightly different periodicity lengths are slowly shifted with respect to each other, the pattern of the superposition (the Moiré pattern, which is locally observable) can move with almost arbitrary speed. In fact the recently discussed phenomenon of superluminal tunneling also consists in the motion of an interference pattern of plane waves (thereby the transmitted wave packet changes its shape, but no information is being transmitted with superluminal speed). A local perturbation of the interference pattern can only propagate with a velocity less than c.

7. Analysis of Zitterbewegung

The Zitterbewegung (that is, the shape of the white curve in the space-time diagram) is described by the expectation value (in the given initial state) of the operator (see, for example, [4])

$$x(t) = \mathrm{e}^{\mathrm{i}H_0 t} x \mathrm{e}^{-\mathrm{i}H_0 t} = x + v_{\mathrm{cl}}(p)\, t + Z(t). \tag{26}$$

Here v_{cl} is the classical velocity operator (24), and $Z(t)$ describes the oscillation,

$$Z(t) = (2\mathrm{i}\, H_0)^{-1}\, (\mathrm{e}^{2\mathrm{i}H_0 t} - \mathbf{1})\, (c\sigma_1 - v_{\mathrm{cl}}(p)). \tag{27}$$

Note that $c\sigma_1$ is the instantaneous velocity according to the Dirac equation, because the time derivative of the position in the Heisenberg picture is just

$$\frac{d}{dt}\, x = \mathrm{i}[H_0, x] = c\,\sigma_1. \tag{28}$$

The initial wave packets (15) and (15) are eigenvectors of σ_1, belonging to the eigenvalue $+1$. Hence the initial velocity of these wave packets is $+c$, hence in Figure 3 and Figure 5, the slope of the white curve at $(x, t) = (0, 0)$ is 45 degrees.

In momentum space, $Z(t)$ is just a multiplication by a matrix-valued function of p. The operator $Z(t)$ anticommutes with H_0. This means that $Z(t)$ maps a state with positive energy onto a state with negative energy. Suppose that $|\psi\rangle$ is, for example, a positive-energy state. Then $Z(t)|\psi\rangle$ has negative-energy and is thus orthogonal to $|\psi\rangle$. Hence, the scalar product $\langle\psi|Z(t)|\psi\rangle$ must be zero. The expectation value $\langle\psi|Z(t)|\psi\rangle$ can only be nonzero for wave packets having both positive- and negative-energy parts.

Wave packets located in different regions of momentum or position space are orthogonal. The operator $Z(t)$ is a multiplication operator in momentum space which means that it does not change the localization properties of a wave packet in momentum space. Hence, Zitterbewegung can only be significant, if the function $\hat{\psi}_{\mathrm{pos}}(p, 0)$ has a significant overlap with the function $\hat{\psi}_{\mathrm{neg}}(p, 0)$ in momentum space.

In position space, the operator $Z(t)$ is nonlocal, but it turns out, that it does not change the approximate localization of a wave packet all too much. Hence, if a wave packet $\psi(x)$ is (approximately) located in some region R of position space, then the function $Z(t)\psi(x)$ is approximately located in a neighborhood of that region. As a consequence, Zitterbewegung is only significant as long as the parts with positive and negative energy are close to each other in position space.

Now, let us consider our examples. Figure 9 shows that the momentum distributions of the positive and negative energy parts of the initial wave packet (15) have a significant overlap in momentum space. This fact does not change with time, because the momentum (and hence the momentum distribution of the initial wave packet) is a conserved quantity according to the free Dirac equation.

Both parts of the initial wave packet have average momenta close to zero, corresponding to average (classical) velocities close to zero. Hence, the corresponding parts will remain close together also in position space. Hence, for this wave packet, Zitterbewegung is a sustained phenomenon. (Actually, the amplitude of the Zitterbewegung vanishes very slowly, as $|t| \to \infty$. A mathematical argument for the asymptotic decay of Zitterbewegung is given in [4].)

For the wave packet in Figure 4, the momentum distributions of the positive and negative energy parts overlap completely (see Figure 10). Hence, we can indeed observe Zitterbewegung, but only as long as the two parts occupy approximately the same region in position space. Because the two parts have opposite velocities, they quickly separate, and the amplitude of the Zitterbewegung decreases rapidly.

For the wave packet in Figure 6, the parts with positive and negative energy move together in position space, but there is no Zitterbewegung, because these parts are located in different regions of momentum space.

8. Conclusion

The paradoxical interference effects occur only for superpositions that are composed of parts with positive and parts with negative energy. The origin for the ripples in the position distribution lies in the fact that the parts with positive and negative energy have opposite phase directions. Because any square-integrable spinor-valued function (in particular, a Gaussian spinor) is, in general, a superposition of positive and negative energies, one is likely to run into these phenomena when one computes a numerical solution of the Dirac equation.

One may argue that superpositions of positive and negative energy states are not physically observable. A wave packet with only one sign of energies shows no Zitterbewegung and behaves reasonably. Unlike this article, a course of relativistic quantum mechanics will certainly put more emphasis on the "reasonabe solutions" than on the paradoxa (see, e.g., [9]). Moreover, in quantum field theory, one tries to construct a Hilbert space of many-particle electron states from the well-behaved positive-energy solutions of the one-particle Dirac equation, and the positronic states are built from charge-conjugated negative-energy solutions. The problem remains that the electronic and the positronic states cannot always be separated in the presence of external fields, and therefore a good understanding of the superposition states is still necessary for investigating the external field problem of QED.

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